20 cm, where soot concentration is highest throughout the flame. It was observed that, with droplets acting as paste, thermocouple wire is covered with a thick soot layer soon after it is introduced into such a region. Values shown by circles at $x = 15$ and 20 cm are those read while the indicator was at rest for a moment just after the thermocouple was introduced into the flame. After that, the readings decrease as the coating of soot grows thicker. Similar effect by soot and droplets was seen also in the case of the suction pyrometer if the cap was removed.

When the suction pyrometer is used, the data at the points lower than 25 cm above the nozzle tip differ, depending on whether the cap is used or not, because soot and/or droplets adhere to the thermocouple when using the suction pyrometer without the cap. Especially at the points of $x = 5$ and 10 cm, where many droplets exist, the difference is remarkable. The large hollows seen in the temperature profiles of the cross sections of $x = 5$ and 10 cm (Fig. 3) may be explained by this fact. Such an apparent temperature drop on the center of the cross section of $x = 5$ cm is evaluated from Fig. 5 as about 350°C.

Conclusion

A technique was developed for detecting droplets within spray combustion flames. Although this technique is inferior to a photographic one in the respect that droplet velocity cannot be measured, it show sufficient reliability on the data giving number and size of droplets, and the device and operation are much simpler. Thus, this technique has the merit that it can be applied directly to the combustion equipment actually used.

The error in temperature measurement due to droplets and soot adhering to themocouple wire is removed by using a suction pyrometer with a small cap. But a suction pyrometer should be applied carefully to the temperature field where the gradient is steep, because it needs a large suction gas flow rate.

References

[1]McCreath, C.G., Roett, M.F., and Chigier, N.A., "A Technique for Measurement of Velocities and Size of Particles in Flames," *Journal of Scientific Instruments*, Vol. 5, June 1972, pp. 601-604.

[2]May, K.R., "The Measurement of Airborne Droplets by the Magnesium Oxide Method," *Journal of Scientific Instruments*, Vol. 27, May 1950, pp. 128-130.

[3]Onuma, Y. and Ogasawara, M., "Studies on the Structure of a Spray Combustion Flame," *15th Sympoium (International) on Combustion*, 1974, The Combustion Institute, pp. 453-465.

Axisymmetric Vibrations of Polar Orthotropic Circular Plates

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Nomenclature

A_1, A_2	= undetermined constants
a	= radius of plate
$D_r, D_\theta, D_{r\theta}$	$= E_r h^3/12,\ E_\theta h^3/12,\ E_{r\theta} h^3/12$
$E_r, E_\theta, E_{r\theta}$	= elastic constants of the plate material
h	= thickness
r, θ	= polar coordinates
W	= lateral deflection
ρ	= mass per unit area
ϕ^2	$= D_\theta / D_r$
μ^2	$= D_{r\theta} / D_r$
λ^2	$= \rho\omega^2 a^4 / D_r$
ω	= circular frequency
σ_r, σ_θ	= stresses in r, θ directions, respectively
$\epsilon_r, \epsilon_\theta$	= strains in r, θ directions, respectively

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I. Introduction

THE axisymmetric vibration of polar orthotropic circular plates has been studied by several authors, namely, Akasaka and Takagishi,[1] Borsuk,[2] and Pandalai and Patel.[3] However, several inconsistencies appear in the literature cited above, as has been pointed out by Leissa.[4] The purpose of this Note is to expose the crux of the problem. The governing equation of motion and boundary conditions are identified from variational principles, and it is seen that there are four boundary conditions. The problem arises because some authors choose to ignore certain boundary conditions or treat them incorrectly. The problem is worked out using the Lagrangian and the Galerkin approach, and this reveals that an error of qualitative nature can arise in the latter case due to violation of a boundary condition at the origin.

II. Theory

The expression given in Ref. 3 for the lowest cyclic frequency of axisymmetric vibration of clamped orthotropic circular plate indicates that this frequency decreases with increase in the value of ϕ^2. Reference 2 also shows a similar behavior in an example for $\phi = 1.4$, where the frequency is lower than that for the isotropic case ($\phi = 1$). However, the expression for frequency given in Ref. 1 shows that the frequency increases with increase in the value of ϕ^2 as is to be physically expected. In all these references, a series solution was used.

In the present investigation, the axisymmetric vibration of a clamped orthotropic circular plate is considered. The stress-strain relations for the case of polar orthotropic material are

$$\sigma_r = E_r\ \epsilon_r + E_{r\theta}\ \epsilon_\theta$$

$$\sigma_\theta = E_{r\theta}\ \epsilon_r + E_\theta\ \epsilon_\theta$$

$$\tau_{r\theta} = G\ \gamma_{r\theta}$$

The Lagrangian function is

$$L = \pi D_r \int_0^a \left(\frac{\rho r W^2{}_{,t}}{D_r} - r W_{,rr}^2 - \frac{\phi^2}{r} W_{,r}^2 - 2\mu^2 W_{,r} W_{,rr} \right) \mathrm{d}r$$

Using the Hamilton's principle and assuming $W(x,t) = W(x)\ e^{i\omega t}$, the equation of motion and the boundary conditions are, respectively,

$$W_{,rrrr} + \frac{2}{r} W_{,rrr} - \frac{\phi^2}{r} \left(\frac{W_{,rr}}{r} - \frac{W_{,r}}{r^2} \right) - \lambda^2 W = 0$$

$$r\ W_{,rr} + \mu^2\ W_{,r} = 0 \quad \text{or} \quad W_{,r} = 0 \quad \text{at } r = 0,\ a$$

$$r\ W_{,rrr} + W_{,rr} - \frac{\phi^2}{r} W_{,r} = 0 \quad \text{or} \quad W = 0 \quad \text{at } r = 0,\ a$$

For a clamped circular plate, the usual assumed geometric conditions are

$$W = W_{,r} = 0 \quad \text{at } r = a \tag{1}$$

It is noted here, that the other two boundary conditions at $r=0$ are of the form

$$\mathop{\mathrm{Lt}}_{r\to 0} \; (r\, W_{,rr} + \mu^2 W_{,r}) = 0 \quad \text{or} \quad W_{,r} = 0 \tag{2a}$$

$$\mathop{\mathrm{Lt}}_{r\to 0} \; (r\, W_{,rrr} + W_{,rr} - \frac{\phi^2}{r} W_{,r}) = 0 \text{ or } W = 0 \tag{2b}$$

Simplifying Eq. (2b) by introducing the result from Eq. (3a) following and using L'Hospital's rule, the boundary conditions reduce to

$$W_{,r} = 0 \quad \text{or} \quad W_{,r} = 0 \tag{3a}$$

$$W_{,rr}(1-\phi^2) = 0 \quad \text{or} \quad W = 0 \tag{3b}$$

For the axisymmetric problem which is considered here, Eq. (3a) is automatically satisfied. From Eq. (3b), since $W \neq 0$, it is noted that the only boundary condition left is

$$(1-\phi^2)\, W_{,rr} = 0 \quad \text{at} \quad r = 0 \tag{4}$$

and that is satisfied automatically for the isotropic case.

Since at $r=0$, $W_{,rr} \neq 0$, Eq. (4) is satisfied only if $\phi^2 = 1$ at the origin. In fact the concept of polar orthotropy cannot be strictly enforced up to the origin which is a singular point. A satisfactory physical explanation is that the radial fibers converging to a point cannot have any cross-sectional area of its own at that point thus leading to isotropic behavior at the origin. It is believed here that this violation of boundary condition leads to erroneous results in certain formulations where differential equations of equilibrium are dealt with as in the case of Galerkin method, but are probably eliminated while dealing with energies in the average sense as in the Lagrangian approach.

III. Examples and Results

The present problem is solved using one term and two term mode-shapes using both the Lagrangian and Galerkin methods, and the results are compared with those available in the literature. The assumed mode shapes are

$$A_1 (1 - \frac{r^2}{a^2})^2$$

and

$$A_1 (1 - \frac{r^2}{a^2})^2 + A_2 (1 - \frac{r^2}{a^2})^3$$

These mode-shapes satisfy the boundary conditions given by Eqs. (1) and (3a) and do not satisfy Eq. (4) except when $\phi^2 = 1$. The results of the present analysis are tabulated in Table 1. When $\phi^2 = 1$ (isotropic case), the Galerkin and Lagrangian solutions are identical, as is to be expected because all the boundary conditions are satisfied. The two-term solution in this case is $\lambda = 10.22$, and this compares excellently with the value of $\lambda = 10.2158$ as given in Ref. 4. However, in the Galerkin method, the frequency decrease with increase in the value of ϕ^2 like those given in Ref. 3. The two-term solution with the Lagrangian approach gives good agreement with those in Ref. 1. The frequency equation for this case is

$$\lambda^4 - (1704 + \frac{1272}{5}\phi^2)\lambda^2 + (115920 + 74592\,\phi^2 + 3024\,\phi^4) = 0$$

To test the argument that it is the nonsatisfaction of Eq. (4) that leads to these erroneous results, a hypothetical problem where the plate is constrained to have $W_{,rr} = 0$ at the origin was examined. Equation (4) is now satisfied, and the solutions obtained by the Lagrangian and Galerkin approaches for an assumed mode shape $W = A_1(1 - 4(r^3/a^3) + 3(r^4/a^4))$ were identical, and the resulting frequency is $\lambda^2 = 18(6+\phi^2)$.

Table 1 Comparison of nondimensional frequency (λ) values

	Present investigation					
	Lagrangian method		Galerkin method		Ref. 1	Ref. 3
ϕ^2	One term	Two term	One term	Two term		
0	8.95	8.42	10.96	11.05	7.16	10.61
1	10.33	10.22	10.33	10.22	10.22	9.8
4	13.66	13.47	8.16	7.56	13.29	7.25
9	17.89	16.69	0	0	16.37	0

IV. Conclusions

The Galerkin and Lagrangian methods of solution give identical results provided the assumed mode-shape satisfies all the boundary conditions obtained from the variational principle. The present analysis shows that the singularity at the origin due to the assumption of general orthotropy, $\phi^2 \neq 1$, could have considerable effect on the results in certain approaches, giving qualitatively erroneous results. However, the Lagrangian approach shows to preserve the behavior qualitatively and gives reasonably good quantitative results as shown here. This is true even though the assumed mode-shape does not satisfy all the boundary conditions arising from the variational principle. It is also expected that in the bending, buckling, and the large deflection analysis of orthotropic circular plates, the Galerkin method will lead to results which are in error qualitatively.

References

[1] Akasaka, T. and Takagishi, T., "Vibration of Corrugated Diaphragm," *JSME Bulletin*, Vol. 1, No. 3, 1958, pp. 215-221.

[2] Borsuk, K., "Free Vibration of Rotations of a Cylindrically Orthotropic Circular Plate," *Archiwum Mechaniki Stosowaneg.*, Vol. 12, No. 5/6, 1960, pp. 649-665.

[3] Pandalai, K. A. V. and Patel, S. A., "Natural Frequencies of Orthotropic Circular Plate," *AIAA Journal*, Vol. 3, April 1965, pp. 780-781.

[4] Leissa, A.W., "Vibration of Plates," NASA SP-160.

Effects of Atomic Oxygen on Graphite Ablation

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METZER et al.[1] derived a semiempirical formula to describe the observed ablation rates of commercial grade (i.e., isotropic) graphites in the form

$$\dot{m}(R/p)^{1/2} = 1.19\times 10^6 e^{-22,140/T_w}\{30.5/R + 4.85\times 10^{15} \times [e^{-22,140/T_w}/(1+1.6\times 10^7 p^{-1/3} e^{-61,700/T_w})]^2\}^{-1/2} \tag{1}$$

where $\dot{m}$, R, p, and T_w are mass loss rate in gcm^{-2} sec^{-1}, nose radius in cm, stagnation-point pressure in atm, and wall temperature in K, respectively. The purpose of the present

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